# Massive complex scalar field in the Kerr-Sen geometry: exact solution of wave equation and Hawking radiation

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The separated radial part of a massive complex scalar wave equation in the Kerr-Sen geometry is shown to satisfy the generalized spheroidal wave equation which is, in fact, a confluent Heun equation up to a multiplier. The Hawking evaporation of scalar particles in the Kerr-Sen black hole background is investigated by the Damour-Ruffini-Sannan's method. It is shown that quantum thermal effect of the Kerr-Sen black hole has the same character as that of the Kerr-Newman black hole.

Key words: Klein-Gordon equation, Generalized spheroidal wave equation, confluent Heun equation, Hawking effect

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## I. INTRODUCTION

In a recent paper, [1] we have investigated exact solution of a massive complex scalar field equation in the Kerr-Newman black hole background, and demonstrated that both its radial part and its angular part can be transformed into the form of a generalized spheroidal wave equation. [2] Previous work on solution of a massive scalar wave equation in the Kerr(-Newman) space-time had been completed in Refs. [3,4]. It is interesting to extend our analysis to solution of a scalar wave equation in a Kerr-Sen black hole background. [5] The Kerr-Sen solution arising in the low energy effective string field theory is a rotating charged black hole generated from the Kerr solution. The thermodynamic property of this twisted Kerr black hole was discussed in Ref. [6] by using separation of the Hamilton-Jacobi equation of a test particle. The aim of this paper is to study some exact solutions to a massive charged scalar wave equation and to find its connection to the confluent Heun equation [7] as well as to investigate quantum thermal effect of scalar particles on the Kerr-Sen space-time.

The paper is organized as follows: In Sec. II, we separate a massive charged scalar field equation on the Kerr-Sen black hole background into the radial and angular parts. Sec. III is devoted to transforming the radial part into a generalized spheroidal wave equation and to relating it to the confluent Heun equation. Then, we investigate quantum thermal effect of scalar particles in the Kerr-Sen space-time in Sec. IV. Finally, we summarize our discussions in the conclusion section.

## II. SEPARATING VARIABLES OF KLEIN-GORDON EQUATION ON THE KERR-SEN BLACK HOLE BACKGROUND

Constructed from the charge neutral rotating (Kerr) black hole solution, the Kerr-Sen solution [5] is an exact classical four dimensional black hole solution in the low energy effective heterotic string field theory. In the Boyer-Lindquist coordinates, the Kerr-Sen metric and

the electromagnetic field vector potential can be rewritten as: [6]

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\Sigma}\Big(dt - a\sin^2\theta d\varphi\Big)^2 + \frac{\sin^2\theta}{\Sigma}\Big[adt - (\Sigma + a^2\sin^2\theta)d\varphi\Big]^2 + \Sigma\Big(\frac{dr^2}{\Delta} + d\theta^2\Big) \\ \mathcal{A} &= \frac{-Qr}{\Sigma}(dt - a\sin^2\theta d\varphi)\,, \end{aligned} \tag{1}$$

where $\Delta = r^2 + 2(b-M)r + a^2 = (r-r_+)(r-r_-)$, $\Sigma = r^2 + 2br + a^2\cos^2\theta$ and $r_\pm = M - b \pm \epsilon$ with $\epsilon = \sqrt{(M-b)^2 - a^2}$.

This metric describes a black hole carrying mass $M$, charge $Q$, angular momentum $J = Ma$, and magnetic dipole moment $Qa$. The twist parameter $b$ is related to the Sen's parameter $\alpha$ via $b = Q^2/2M = M\tanh^2(\alpha/2)$. Because $M \geq b \geq 0$, $r = r_-$ is a new singularity in the region $r \leq 0$, the event horizon of the Kerr-Sen black hole is located at $r = r_+$. The area of the outer event horizon of the twisted Kerr solution [5] is given by $A_+ = 4\pi(r_+^2 + 2br_+ + a^2) = 8\pi Mr_+$.

We consider the solution of a massive charged test scalar field on the Kerr-Sen black hole background (We use Planck unit system $G = \hbar = c = k_B = 1$ throughout the paper). Because the Kerr-Sen metric (1) only differs from the Kerr(-Newman) solution by the form of two functions $\Delta$ and $\Sigma$, the Klein-Gordon field equation satisfied by the complex scalar wave function $\Phi$ with mass $\mu$ and charge $q$ in such a space-time can be separated as $\Phi(t,r,\theta,\varphi) = R(r)S_{m,0}^{\ell}(ka,\theta)e^{i(m\varphi-\omega t)}$, in which the angular part $S_{m,0}^{\ell}(ka,\theta)$ is an ordinary spheroidal angular wave function with spin-weight $s = 0$, while the radial part can be given as follows:

$$\partial_r[\Delta\partial_r R(r)] + \Big[\frac{(Ar - ma)^2}{\Delta} + k^2\Delta + 2Dr - \lambda\Big]R(r) = 0\,, \tag{2}$$

here $\lambda$ is a separation constant, $A = 2M\omega - qQ$, $D = A\omega - M\mu^2$, $k = \sqrt{\omega^2 - \mu^2}$ (We assume that $\omega > \mu$). For later convenience, we also denote $\epsilon B = A(M-b) - ma$ and introduce $W_\pm = (A \pm B)/2$.

With further substitution $R(r) = (r-r_+)^{i(A+B)/2}(r-r_-)^{i(A-B)/2}F(r)$, we can transform Eq. (2) for $R(r)$ into a modified generalized spheroidal wave equation with imaginary spin-weight $iA$ and boost-weight $iB$ for $F(r)$: [1]

$$\Delta\partial_r^2F(r) + 2[i\epsilon B + (1+iA)(r - M + b)]\partial_r F(r) + [k^2\Delta + 2Dr + iA - \lambda]F(r) = 0\,. \tag{3}$$

Eq. (2) has two regular singular points $r = r_\pm$ with indices $\pm iW_+$ and $\pm iW_-$, respectively, whereas Eq. (3) has indices $\rho_+ = 0, -2iW_+$ and $\rho_- = 0, -2iW_-$ at two singularities $r = r_\pm$, respectively. The infinity is an irregular singularity of Eqs. (2) and (3). Eq. (3) has the same form as the radial part of the massive complex scalar wave equation in the Kerr-Newman geometry [1] with its solution (when $\mu = 0$) named as the generalized spheroidal wave function. [2] It is interesting to note that a special solution of function $F(r)$ satisfies the Jacobi equation of imaginary index when $\omega = \pm\mu = qQ/M$ (namely, $k = D = 0$).

## III. GENERALIZED SPHEROIDAL WAVE FUNCTION AND HEUN EQUATION

In this section, we shall show that the generalized spheroidal wave equation (3) of imaginary number order is, in fact, a confluent form of Heun equation. [7] To this end, let us make a coordinate transformation $r = M - b + \epsilon z$ and substitute $R(r) = (z-1)^{i(A+B)/2}(z+1)^{i(A-B)/2}F(z)$ into Eq. (2), then we can reduce it to the following standard forms of a generalized spheroidal wave equation: [1,7]

$$(z^2-1)R''(z) + 2zR'(z) + \Big[(\epsilon k)^2(z^2-1) + 2D\epsilon z + \frac{(Az+B)^2}{z^2-1} + 2D(M-b) - \lambda\Big]R(z) = 0\,, \tag{4}$$

and

$$(z^2-1)F''(z) + 2[iB + (1+iA)z]F'(z) + [(\epsilon k)^2(z^2-1) + 2D\epsilon z + 2D(M-b) + iA - \lambda]F(z) = 0\,, \tag{5}$$

where a prime denote the derivative with respect to its argument.

The spin-weighted spheroidal wave function $F(z)$ is symmetric under the reflect $k \to -k$. Letting $F(z) = e^{i\epsilon kz}G(z)$ without loss of generality, we can transform Eq. (5) to

$$(z^2-1)G''(z) + 2[iB + (1+iA)z + i\epsilon k(z^2-1)]G'(z) + [2i\epsilon k(1+iA-iD/k)z - 2\epsilon kB + iA + 2D(M-b) - \lambda]G(z) = 0\,. \tag{6}$$

By means of changing variable $z = 1 - 2x$, we arrange the singularities $r = r_+$ $(z = 1)$ to $x = 0$ and $r = r_-$ $(z = -1)$ to $x = 1$, respectively, and reduce Eq. (6) to a confluent form of Heun's equation [7,8]

$$G''(x) + \Big(\beta + \frac{\gamma}{x} + \frac{\delta}{x-1}\Big)G'(x) + \frac{\alpha\beta x - h}{x(x-1)}G(x) = 0\,, \tag{7}$$

with $\gamma = 1 + 2iW_+$, $\delta = 1 + 2iW_-$, $\beta = 4i\epsilon k$, $\alpha = -(1 + iA) + iD/k$, $h = \lambda - 2i\epsilon k - iA + 4\epsilon k W_+ - 2Dr_+$.

This confluent Heun equation (7), with $h$ its accessory parameter, has two regular singular points at $x = 0, 1$ with exponents $(0, 1-\gamma)$ and $(0, 1-\delta)$,respectively, as well as an irregular singularity at the infinity point. The power series solution in the vicinity of the point $x = 0$ for Eq. (7) can be written as

$$G(\alpha, \beta, \gamma, \delta, h; x) = \sum_{n=0}^{\infty} g_n x^n\,, \tag{8}$$

and the coefficient $g_n$ satisfies a three-term recurrence relation [7,8]

$$\begin{aligned} &g_0 = 1\,, \quad g_1 = -h/\gamma\,, \\ &(n+1)(n+\gamma)g_{n+1} - \beta(n-1+\alpha)g_{n-1} = [n(n-1-\beta+\gamma+\delta) - h]g_n\,. \end{aligned} \tag{9}$$

It is not difficult to deduce the exponent $1-\gamma$ solution [8] for $x = 0$ and obtain the power series solution in the vicinity of the point $x = 1$ by a linear transformation interchanging the regular singular points $x = 0$ and $x = 1$: $x \to 1 - x$. Expansion of solutions to the confluent Heun's equation in terms of hypergeometric and confluent hypergeometric functions has been presented in Refs. [2,7]. The confluent Heun's functions can be normalized to constitute a group of orthogonal complete functions. [7] It should be noted that Heun's confluent equation also admits quasi-polynomial solutions for particular values of the parameters. [7,8] It follows from the three-term recurrence relation that $G(\alpha, \beta, \gamma, \delta, h; x)$ is a polynomial solution if

$$\begin{aligned} &\alpha = -N\,, \quad \text{with integer} \quad N \geq 0\,, \\ &g_{N+1}(h) = 0\,, \end{aligned} \tag{10}$$

where $g_{N+1}$ being a polynomial of degree $N+1$ in $h$, that is, there are $N+1$ eigenvalues $h_i$ for $h$ such that $g_{N+1}(h_i) \equiv 0$.

## IV. HAWKING RADIATION OF SCALAR PARTICLES

Now we investigate the Hawking evaporation [9] of scalar particles in the Kerr-Sen black hole by using the Damour-Ruffini-Sannan's (DRS) method. [10] This approach only requires the existence of a future horizon and is completely independent of any dynamical details of the process leading to the formation of this horizon. The DRS method assumes analyticity properties of the wave function in the complexified manifold.

In the following, we shall consider a wave outgoing from the event horizon $r_+$ over interval $r_+ < r < \infty$. According to the DRS method, a correct outgoing wave $\Phi^{\rm out} = \Phi^{\rm out}(t, r, \theta, \varphi)$ is an adequate superposition of functions $\Phi^{\rm out}_{r>r_+}$ and $\Phi^{\rm out}_{r<r_+}$:

$$\Phi^{\rm out} = C[\eta(r-r_+)\Phi^{\rm out}_{r>r_+} + \eta(r_+-r)\Phi^{\rm out}_{r<r_+}e^{2\pi W_+}] \, , \tag{11}$$

where $\eta$ is the conventional unit step function, $C$ is a normalization factor.

In fact, components $\Phi^{\rm out}_{r>r_+}$ and $\Phi^{\rm out}_{r<r_+}$ have asymptotic behaviors:

$$\Phi^{\rm out}_{r>r_+} = \Phi^{\rm out}_{r>r_+}(t, r, \theta, \varphi) \longrightarrow c_1(r-r_+)^{iW_+}S^{\ell}_{m,0}(ka, \theta)e^{i(m\varphi-\omega t)} \, , \quad (r \rightarrow r_+) \tag{12}$$

$$\Phi^{\rm out}_{r<r_+} = \Phi^{\rm out}_{r<r_+}(t, r, \theta, \varphi) \longrightarrow c_2(r-r_+)^{-iW_+}S^{\ell}_{m,0}(ka, \theta)e^{i(m\varphi-\omega t)} \, , \quad (r \rightarrow r_+) \tag{13}$$

when $r \rightarrow r_+$. Clearly, the outgoing wave $\Phi^{\rm out}_{r>r_+}$ can't be directly extended from $r_+ < r < \infty$ to $r_- < r < r_+$, but it can be analytically continued to an outgoing wave $\Phi^{\rm out}_{r<r_+}$ that inside event horizon $r_+$ by the lower half complex $r$-plane around unit circle $r = r_+ - i0$:

$$r - r_+ \longrightarrow (r_+ - r)e^{-i\pi} \, .$$

By this analytical treatment, we have

$$\Phi^{\rm out}_{r<r_+} \sim c_2(r-r_+)^{-iW_+}S^{\ell}_{m,0}(ka, \theta)e^{i(m\varphi-\omega t)} \, . \tag{14}$$

Eq. (13) just takes one solution to the radial equation inside the event horizon $r_+$, it has the same form of Eq. (14) generated by the analytical method. As $\Phi^{\rm out}_{r>r_+}$ differs $\Phi^{\rm out}_{r<r_+}$ by a factor $(r-r_+)^{-2iW_+}$, then a difference factor $e^{2\pi W_+}$ emerges due to the above analytical treatment. Thus we can derive the relative scattering probability of the scalar wave at the event horizon

$$\Big|\frac{\Phi^{\rm out}_{r>r_+}}{\Phi^{\rm out}_{r<r_+}}\Big|^2 = e^{-4\pi W_+}\,, \tag{15}$$

and obtain the thermal radiation spectrum with the Hawking temperature $T = \kappa/2\pi$.

$$\langle \mathcal{N} \rangle = |C|^2 = \frac{1}{e^{4\pi W_+} - 1}\,, \qquad W_+ = \frac{Ar_+ - ma}{2\epsilon} = \frac{\omega - m\Omega - q\Phi}{2\kappa}\,, \tag{16}$$

where the angular velocity at the horizon is $\Omega = a/2Mr_+$, the electric potential is $\Phi = Q/2M = b/Q$, the surface gravity at the pole is $\kappa = (r_+ - M + b)/2Mr_+ = \epsilon/2Mr_+$.

The black body radiation spectrum (16) demonstrates that the thermal property of Kerr-Sen black hole is similar to that of Kerr-Newman black hole though its geometry character is like that of the Kerr solution. [6] Correspondingly, there exist four thermodynamical laws of the Kerr-Sen black hole, similar to those of Kerr-Newman black hole thermodynamics.

## V. CONCLUSION

In this paper, we have shown that the separation of variables of the scalar wave equation in the Kerr-Newman black hole background can apply completely to the case of the twisted Kerr solution. The separated radial part can be recast into the generalized spheroidal wave equation, which is, in fact, a confluent form of Heun equation.

In addition, we find that the thermal property of the twisted Kerr black hole resembles that of Kerr-Newman black hole though its geometry character is like that of the Kerr solution. The Kerr-Sen solution shares similar four black hole thermodynamical laws and quantum thermal effect as the Kerr-Newman space-time does.

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